Universal measurability and summable families in tvs

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ABSTRACT

The main result of this paper, in a somewhat specialized form, is as follows. Let α, β be two Hausdorff linear topologies on X such that (X, β) is complete, $\alpha \subset \beta$ and the identity $j:(X, \alpha) \to (X, \beta)$ is universally measurable. Then the subseries convergent series are the same for α and β.

INTRODUCTION

The purpose of this paper is to prove the following theorem conjectured by E. Thomas in [8].

Let α, β be two linear Hausdorff topologies on X such that (X, β) is complete, $\alpha \subset \beta$ and

(UM) The identity j: $(X, \alpha) \to (X, \beta)$ is universally measurable.
Then any α-subseries convergent series is β-subseries convergent.

In fact, the method we use can be applied with no countability restrictions and our main theorem is stated for subfamily summable families. In the setting of complete topological vector spaces this result provides the strongest Orlicz-Pettis type theorem obtained up to now. The proof relies essentially on recent investigations concerning the so called discrete copies of rings of sets and their relevance to exhaustivity and Orlicz-Pettis problem [4] [5]. Also, as a byproduct we obtain that universally measurable linear maps on F-spaces are continuous – a version of a classic theorem of Banach.

§ 1. PRELIMINARIES

1.1. In this paper Γ is an infinite set, $\mathscr{P}(\Gamma)$ denotes the power set of Γ, $\mathscr{F}(\Gamma)$ is the family of finite subsets of Γ. For $A, B \in \mathscr{P}(\Gamma)$, $A \Delta B = (A \backslash B) \cup (B \backslash A)$. $\mathscr{P}(\Gamma)$ with the symmetric difference Δ as addition is a commutative group.

$P(\Gamma) = \{0, 1\}^{\Gamma}$ denotes the topological product of Γ copies of discrete two point space $\{0, 1\}$. The addition $+$ in $P(\Gamma)$ is generated by the addition modulo 2 in $\{0, 1\}$. $P(\Gamma)$ is a compact commutative group. We denote by $a, b, c, \ldots$ elements of $P(\Gamma)$; $a^{\gamma} = \{0, 0, \ldots pr_{\gamma}\, a, 0, 0, \ldots)$, where pr_{γ} is the γ-projection; $e = (1, 1, \ldots)$. $P(\Gamma)$ will also be denoted P if no ambiguity about Γ can arise.

A subset of Γ being represented by its characteristic function on Γ, $\mathscr{P}(\Gamma)$ may be identified in a natural way with $P(\Gamma)$. The symmetric difference corresponds then to the addition $+$. Via this identification the operations corresponding to $\cup, \backslash, \cap$ in $\mathscr{P}(\Gamma)$ may also be defined in $P(\Gamma)$. They will be denoted $\vee, -, \cdot$ in order to avoid ambiguity with the set theoretical operations which can be performed on subsets of $P(\Gamma)$ as well. Given $a \in P(\Gamma)$ the support of a, $\operatorname{supp} a = \{\gamma \in \Gamma : pr_{\gamma}\, a = 1\}$. $F(\Gamma)$ denotes the set of elements with finite supports i.e., the set corresponding to $\mathscr{F}(\Gamma)$.

We define a measure on $\{0, 1\}$ by setting masses 1/2 at each point. The corresponding product measure on $P(\Gamma)$ will be denoted λ.

1.2. Let X be a topological space and T a Hausdorff topological space. A function $f : T \to X$ is *universally measurable* if it is measurable with respect to any finite Radon measure on T. Let X be a topological vector space (*tvs*); f is *semi-norm universally measurable* if there exists a family Q of F-semi-norms defining the topology of X and such that $f : T \to (X, q)$ is universally measurable for any $q \in Q$.

A universally measurable map is semi-norm universally measurable for any family of continuous F-semi-norms on X. If X is hereditary Lindelöf both notions coincide.

For image measures, Lusin measurable maps etc ... we follow the terminology of [7]. Note, however, that "μ-measurable" in this paper = "Borel μ-measurable" in [7].

Let S, T be two abstract sets. Following the terminology of [6], let a measure $(S, \mathscr{B}, \mu)$ be given and let g be a map from S into T. The measure $(T, \mathscr{B}^g, \mu^g)$ induced by g on T is defined by

$$E \in \mathscr{B}^g \Leftrightarrow g^{-1}(E) \in \mathscr{B}, \;\; \mu^g(E) = \mu(f^{-1}(E)).$$

The following result will be needed below.

1.3. PROPOSITION. *Let T, Y be topological spaces, T being Hausdorff. Let $f : T \to Y$ be a universally measurable map, $g : P(\mathbf{N}) \to T$ a universally Lusin measurable [in particular – continuous] map. Then fg is universally measurable.*

PROOF. Given μ, by ([7], p. 32, Th. 8, Case 1), it follows that the image measure $g\mu$ on T is precisely the measure μ^g induced by g on T. Now, by assumption f is $g\mu$-measurable $=\mu^g$-measurable and this implies that fg is μ-measurable.

1.4. Let X be a *tvs*. We will be interested in the behaviour of maps on $P(\Gamma)$ into X which correspond to additive set functions on $\mathscr{P}(\Gamma)$ and therefore the terminology of vector measure theory will be used. For instance, $\boldsymbol{m}: P(\Gamma) \to X$ is *finitely additive* if for $a, b \in P(\Gamma)$ with $a \cdot b = 0$ $\boldsymbol{m}(a+b) = \boldsymbol{m}(a) + \boldsymbol{m}(b)$; $\boldsymbol{m}$ is *exhaustive* if for any sequence (a_n) of *disjoint* (i.e., $a_n \cdot a_m = 0$ for $n \neq m$) elements in $P(\Gamma)$, $\boldsymbol{m}(a_n) \to 0$ etc . . . We stress that "finitely additive" does not mean "additive" in the sense of the group $\{0, 1\}^\Gamma$ because $\boldsymbol{m}$ is in general not a homomorphism on $\{0, 1\}^\Gamma$.

A family $(x_\gamma: \gamma \in \Gamma)$ of elements of X is said to be *subfamily summable* (an *sbs* family) if for any subset $\Delta \subset \Gamma$ the corresponding family $(x_\gamma: \gamma \in \Delta)$ is summable in X.

Let X be Hausdorff and let an *sbs* family $(x_\gamma: \gamma \in \Gamma)$ be given. Define an additive set function $\boldsymbol{m}: \mathscr{P}(\Gamma) \to X$ by

$$\boldsymbol{m}(\Delta) = \sum_{\gamma \in \Delta} x_\gamma, \ \Delta \in \mathscr{P}(\Gamma).$$

It is clear that $\boldsymbol{m}$ may be treated as a finitely additive map on $P(\Gamma)$ and that

$$\boldsymbol{m}: P(\Gamma) \to X \text{ is continuous.}$$

Conversely, if $\boldsymbol{m}: P(\Gamma) \to X$ is a continuous finitely additive map then $(\boldsymbol{m}(e^\gamma): \gamma \in \Gamma)$ is an *sbs* family in X and for any $d \in P(\Gamma)$

$$\boldsymbol{m}(d) = \sum_{\gamma \in \Delta} \boldsymbol{m}(e^\gamma), \ \Delta = \operatorname{supp} d.$$

Throughout this paper we shall constantly identify finitely additive maps $\boldsymbol{m}$ on $P(\Gamma)$ with additive set functions on $\mathscr{P}(\Gamma)$. Having this in mind, we note the following facts.

1.5. Let X be a Hausdorff *tvs*. The following are equivalent.

(1) $\boldsymbol{m}: P(\Gamma) \to X$ is continuous.

(2) $\boldsymbol{m}: \mathscr{P}(\Gamma) \to X$ is almost $\aleph_0$-concentrated on Γ [5].

(3) $(\boldsymbol{m}(e^\gamma): \gamma \in \Gamma)$ is an *sbs* family.

1.6. Let $\mathbf{N} = \{1, 2, \ldots\}$. In this special case $\{0, 1\}^{\mathbf{N}}$ is compact metrizable and every Borel measure on $\{0, 1\}^{\mathbf{N}}$ is Radon.

A sequence (x_n) of elements of X is said to be *subseries convergent* (*sbc*) if for any subsequence $(n_k) \subset \mathbf{N}$ the series $\sum_{k=1}^\infty x_{n_k}$ is convergent. The fact that, given a sequence $(x_n: n \in \mathbf{N})$ in X, it is subfamily summable iff it is subseries convergent is classic and due to Orlicz. Therefore 1.5 admits in this case the following stronger form:

1.7. The following are equivalent.

(1) $\mathbf{m}: P(\mathbf{N}) \to X$ is continuous.

(2) $\mathbf{m}: \mathscr{P}(\mathbf{N}) \to X$ is countably additive.

(3) $(\mathbf{m}(e^n): n \in \mathbf{N})$ is subseries convergent.

1.8. A finitely additive map $\mathbf{m}: F(\Gamma) \to X$ is said to be an *isomorphism* (cf. [5]) if there exists a 0-*nbhd* W in X such that for every $a, b \in F(\Gamma)$

$$a \neq b \Rightarrow \mathbf{m}(a) - \mathbf{m}(b) \notin W.$$

§ 2. PREPARATORY LEMMAS

Throughout this section $\mathbf{m}$ is a finitely additive map.

2.1. LEMMA. *Let X be a tvs. If $\mathbf{m}: P(\mathbf{N}) \to X$ is λ-measurable then the sequence $\{\mathbf{m}(e^n): n \in \mathbf{N}\}$ is a bounded set in X.*

PROOF. If it is not so, there is a balanced 0-*nbhd* W in X such that for each $n \in \mathbf{N}$ there exists $k_n \in \mathbf{N}$ for which

$$\mathbf{m}(e^{k_n}) \notin nW.$$

Let V be an open 0-*nbhd* such that $V - V \subset W$ and denote $A_n = \mathbf{m}^{-1}\{nV\}$. Since V is absorbing, $\lambda(A_n) \to 1$ as $n \to \infty$. Take A_n such that $\lambda(A_n) > 1/2$. Represent $P(\mathbf{N})$ in the form $K \times \{0, 1\}$ where $K = \{0, 1\}^{\mathbf{N} \setminus \{k_n\}}$.

Consider K-sections of A_n

$$B_a = (a \times \{0, 1\}) \cap A_n, \quad a \in K.$$

If $b, b' \in B_a$ then $nW \supset nV - nV \ni \mathbf{m}(b) - \mathbf{m}(b') = \mathbf{m}(a, \delta) - \mathbf{m}(a, \delta')$ where δ and δ' take values 0 or 1. If $\delta - \delta' = \pm 1$, $\mathbf{m}(b) - \mathbf{m}(b') = \pm \mathbf{m}(e^{k_n}) \notin nW$, which is impossible. Hence $\delta - \delta' = 0$ and consequently B_a has measure $1/2$. It follows by Fubini's theorem that $\lambda(A_n) \leqslant 1/2$, a contradiction.

2.2. PROPOSITION. *Let X be a tvs. A universally measurable $\mathbf{m}: P(\Gamma) \to X$ is disjointly bounded (i.e., for any sequence of disjoint elements $(a_n) \subset P$ the set $\{\mathbf{m}(a_n): n \in \mathbf{N}\}$ is bounded in X).*

PROOF. Define $\varkappa: P(\mathbf{N}) \to P(\Gamma)$ by

$$\varkappa(d) = \bigvee_{n \in D} a_n \text{ where } D = \operatorname{supp} d.$$

If $\{\mathbf{m}(a_n): n \in \mathbf{N}\}$ is not bounded, then the map $\mathbf{n} = \mathbf{m} \cdot \varkappa: P(\mathbf{N}) \to X$ cannot be λ-measurable by 2.1. On the other hand, as $\varkappa$ is continuous, $\mathbf{m} \cdot \varkappa$ is universally measurable by 1.3. A contradiction.

Since for additive set functions on $\mathscr{F}(\mathbf{N})$ disjoint boundedness and boundedness of the range coincide, we have

2.3. COROLLARY. *If* $\boldsymbol{m}: P(\mathrm{N}) \to X$ *is universally measurable then* $\boldsymbol{m}\{F(\mathrm{N})\}$ *is a bounded set in* X.

2.4. LEMMA. *Assume* X *is metrizable. Let* $\boldsymbol{m}: P(\mathrm{N}) \to X$ *be such that* $\boldsymbol{m}|F(\mathrm{N})$ *is an isomorphism* [5]. *Then* $\boldsymbol{m}$ *is not* λ*-measurable.*

PROOF. Assume $\boldsymbol{m}$ is λ-measurable. By a result of Sazonov ([6], Cor. 1 and 2 of Th. 11) $\boldsymbol{m}$ is λ-essentially separably valued. We can find $(x_m) \subset X$ such that $x_m + V = V_m$, $m \in \mathrm{N}$, cover $\boldsymbol{m}(P \backslash A_0)$. Hence A_0, $A_m = \boldsymbol{m}^{-1}(V_m)$ cover P and for some $m \in \mathrm{N}$ $\lambda(A_m) = \theta > 0$. We shall show that $\lambda(A_m) < \theta$ and this contradiction will prove the lemma.

Let $n \in \mathrm{N}$ be so large that $2^{-n} < \theta$ and represent P in the form $P = K \times L$ where $L = \{0, 1\}^{\{1, 2, \ldots n\}}$. Consider K-sections of A_m

$$B_a = (a \times L) \cap A_m, \ a \in K.$$

If $d, d' \in B_a$ then $\boldsymbol{m}(d) - \boldsymbol{m}(d') \in V - V \subset W$. On the other hand for some $b_1, b_2 \in L$ we have $\boldsymbol{m}(d) - \boldsymbol{m}(d') \equiv \boldsymbol{m}(a, b_1) - \boldsymbol{m}(a, b_2) = \boldsymbol{m}(0, b_1) - \boldsymbol{m}(0, b_2)$. As we assumed that $\boldsymbol{m}|F$ is an isomorphism $b_1 \neq b_2$ is impossible, because we would have $\boldsymbol{m}(0, b_1) - \boldsymbol{m}(0, b_2) \notin W$. Hence $b_1 = b_2$ and all K-Sections of A_m have measure $1/2^n$. It follows by Fubini's theorem that $\lambda(A_m) \leqslant 1/2^n < \theta$.

2.5. LEMMA. *Assume* X *is metrizable. If* $\boldsymbol{m}: P(\mathrm{N}) \to X$ *is universally measurable then* $\boldsymbol{m}|F(\mathrm{N})$ *is exhaustive.*

PROOF. By 2.3 $\boldsymbol{m}|F(\mathrm{N})$ has bounded range. If $\boldsymbol{m}$ is not exhaustive on $F(\mathrm{N})$, then we can find a disjoint sequence $(a_n) \subset F(\mathrm{N})$ and a 0-*nbhd* V such that $\boldsymbol{m}(a_n) \notin V$ for $n \in \mathrm{N}$. Defining $\varkappa$ as in the proof of 2.2 we have

$$\boldsymbol{n} = \boldsymbol{m} \cdot \varkappa : P(\mathrm{N}) \to X$$

such that

i) $\boldsymbol{n}|F(\mathrm{N})$ is bounded;

ii) $\boldsymbol{n}(e^n) \notin V$ for all $n \in \mathrm{N}$.

Applying ([4] Th. 3) there is an infinite subset $\boldsymbol{M} \subset \mathrm{N}$ such that $\boldsymbol{n}|F(\boldsymbol{M})$ is an isomorphism. Denote the canonical injection of $P(M)$ into $P(\mathrm{N})$ by ν and consider $\boldsymbol{k} = \boldsymbol{m} \cdot \varkappa \cdot \nu : P(\boldsymbol{M}) \to X$. We have $\boldsymbol{k}|F(\boldsymbol{M}) \equiv \boldsymbol{n}|F(\boldsymbol{M})$, hence $\boldsymbol{k}$ is not λ-measurable by 2.4. A contradiction with 1.3.

2.6. LEMMA. *Let* X *be metrizable. If* $\boldsymbol{m}: P(\Gamma) \to X$ *is universally measurable then* $\boldsymbol{m}$ *is exhaustive.*

PROOF. Take a disjoint sequence $(a_n) \subset P(\Gamma)$ and define $\varkappa : P(\mathrm{N}) \to P(\Gamma)$ by the usual formula

$$\varkappa(d) = \bigvee_{n \in \Delta} a_n, \text{ where } \Delta = \operatorname{supp} d.$$

Then $\boldsymbol{m}\cdot\varkappa\colon P(\mathrm{N}) \to X$ is universally measurable by 1.3 and 1.5. Consequently, $(\boldsymbol{m}\varkappa)(e^n) = \boldsymbol{m}(a_n) \to 0$ by 2.5.

§ 3. MAIN RESULTS

3.1. THEOREM. *Let X be a topological vector space, $\boldsymbol{m}\colon P(\Gamma) \to X$ a finitely additive map. If $\boldsymbol{m}$ is semi-norm universally measurable then $\boldsymbol{m}$ is exhaustive.*

PROOF. Let Q be a family of F-semi-norms defining the topology of X and such that for $q \in Q$ $\boldsymbol{m}\colon P(\Gamma) \to (X, q)$ is universally measurable. Let X_q be the quotient F-normed space associated with q and let r be the corresponding quotient map. Then $r\cdot\boldsymbol{m}\colon P(\Gamma) \to X_q$ is universally measurable and we may apply 2.6. This implies the theorem.

With the same notation we have the following corollary.

3.2. COROLLARY. *Let X be Hausdorff complete and assume that for some Hausdorff linear topology α weaker than the original one on X $\boldsymbol{m}\colon P(\Gamma) \to (X, \alpha)$ is almost $\mathscr{M}$-concentrated (see [5], $\mathscr{M} = \operatorname{card}\Gamma$) on Γ. Then $\boldsymbol{m}$ is continuous.*

PROOF. Indeed, as $\boldsymbol{m}$ is exhaustive, we can define $\boldsymbol{n}$ on $P(\Gamma)$ by putting $\boldsymbol{m}(d) = \sum_{\gamma\in\Delta} \boldsymbol{m}(e^\gamma)$ where $\Delta = \operatorname{supp} d$ and the summation is in the original topology of X. Then $\boldsymbol{m} - \boldsymbol{n}$ into (X, α) is simultaneously purely $\mathfrak{m}$-exhaustive and almost $\mathfrak{m}$-concentrated on Γ (see [5]), hence identically zero. The result now follows by 1.5.

3.3. THEOREM. *Let α, β be two Hausdorff linear topologies on X such that $\alpha \subset \beta$, (X, β) is complete and*

(UM) *The identity $j\colon (X, \alpha) \to (X, \beta)$ is semi-norm universally measurable.*

If a finitely additive map $\boldsymbol{m}\colon P(\Gamma) \to (X, \alpha)$ is continuous then it is β-continuous. In particular, the subfamily summable families are the same for α and β.

In the countable case 3.3 takes stronger form.

3.4. COROLLARY. *Let α, β be two Hausdorff linear topologies on X such that $\alpha \subset \beta$, (X, β) is sequentially complete and (UM) is satisfied. Then the subseries convergent series are the same for α and β.*

PROOF OF THE THEOREM. By 1.5 and 1.3 the first statement follows from 3.2. For the second, having an *sbs* family $(x_\gamma\colon \gamma \in \Gamma)$ in (X, α), we define a continuous finitely additive $\boldsymbol{m}\colon P(\Gamma) \to (X, \alpha)$ in the usual way (see § 1) and apply the first part together with 1.5.

REMARKS. 1) Corollary 3.4 settles in the positive the conjecture of Thomas ([8] p. 54) in the category of topological vector spaces. Some results of this type are also valid in groups (for instance the topology α could be assumed merely a group topology in 3.2–3.4), but this more general case seems to need a separate treatment. We hope to give an account of our investigations in this direction elsewhere.

2) In the setting of linear topologies our 3.4 above is stronger than Th. 3 in [1] (but we need sequential completeness of β). On the other hand the main theorem of [1] (Th. 1) under a stronger assumption (= Borel measurability) gives a stronger statement. Such a result is not possible in our approach in view of a counter-example in [3].

3) Let X be locally convex or, more generally, assume that X can be embedded in a product of locally bounded F-spaces. In this case the fact that a universally measurable $\boldsymbol{m}: P(\Gamma) \to X$ is exhaustive may be proved without using Sazonov's results. Indeed by assumption on X, $\boldsymbol{m}$ being disjointly bounded by 2.2, is bounded. If $\boldsymbol{m}$ would not be exhaustive we could find a countable subset Δ in Γ such that $\boldsymbol{m}|P(\Delta)$ is an isomorphism (see [5]) by applying ([4], Th. 3). Such an isomorphism cannot be λ-measurable and this implies that $\boldsymbol{m}$ is not universally measurable in $P(\Gamma)$.

3.5. COROLLARY. *Let X be a K-analytic* [7] *complete tvs. Subfamily summable families coincide for all Hausdorff linear topologies weaker than the original one on X.*

PROOF. Apply 3.3 and [7] p. 126, Cor. 1. Another consequence of our results is the following version of a well-known theorem of Banach ([2] p. 23, Th. 4).

3.6. THEOREM. *Let E be an F-space, X a tvs and $f: E \to X$ a universally measurable map. Then f is continuous.*

PROOF. If f is not continuous, then it is possible to find $(y_n) \subset E$, $y_n \to 0$ such that $f(y_n)$ is not bounded. Passing to a subsequence we may assume that $\sum_{n=1}^{\infty} y_n$ is subseries convergent. Defining $\boldsymbol{m}: P(\mathrm{N}) \to E$ in the usual way, $f \cdot \boldsymbol{m}: P(\mathrm{N}) \to X$ is universally measurable by 1.3. Hence $f \cdot m$ is exhaustive, so $f \cdot \boldsymbol{m}(e^n) = f(y_n) \to 0$; a contradiction.

REMARK 4. E may be taken with the Hausdorff linear topology which is the finest linear topology making continuous linear maps $f_i: E_i \to E$ where $(E_i): i \in I)$ is certain family of F-spaces. It is also sufficient to assume f semi-norm universally measurable.

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